An Important Properties of Entanglement: Pairwise Entanglement can Only be Transferred by Entangled Pair

Xiao-Qiang Xi[1,2], W. M. Liu[2]
[1]*Department of Applied Mathematics and Physics, Xi'an Institute of Posts and Telecommunications, Xi'an 710061, China and*
[2]*Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100080, China*
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Basing on the calculation of all the pairwise entanglement in the n ($n \leq 6$)-qubit Heisenberg XX open chain with system impurity, we find an important result: pairwise entanglement can only be transferred through entangled pair. The non-nearest pairwise entanglement will has the possibility to exist as long as there has even number qubit in their middle. This point means that we can get longer distance entanglement in solid system.

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It is well known that entanglement has some prominent applications: firstly, it can be used to test some fundamental questions of the quantum mechanics; secondly, it is the key ingredient of the quantum information processing such as quantum teleportation, superdense cording, quantum computation, quantum communication, quantum computational speed-ups and quantum cryptographic protocols; thirdly, the entangled states can be used to the sensitivity of interferometric measurements such as quantum lithography [1], quantum optical gyroscope [2], quantum clock synchronization and positioning [3] and frequency metrology [4]; fourthly is that entanglement play a central role in the study of strongly correlated quantum systems [5], ground-state entanglement is help us to understand the quantum phase transition [6], Mott insulator-superfluid transition and quantum magnet-paramagnet transition.

In all the implications of entanglement, the most important thing is to find entangled pair. The photons [7], the energy level of different trapped ions [8] (or different atoms [9]), qubit in crystal lattices [10], qubit in Josephson junctions [11] and Bose-Einstein condensates [12] are often used to produce entanglement, but their complicated equipment will prevent them from using cosmically. While using spin chains [13] to produce entanglement seems more convenient, because it can be scaled easily and its equipment will be very simple, many interesting work are developed around the entanglement in spin chain(see Ref. [14] and their references), while most of them [15, 16] are concerned with the nearest pairwise entanglement, that is important but far from perfect. The best entanglement should be the non-nearest pairwise, for example, in a three qubits Heisenberg XX open chain, the entanglement between the first and the third qubits (if exist) is more practical than the first and the second because the former has farther entanglement distance.

If we can find the law of the non-nearest pairwise entanglement in spin chain, then we have the possibility to construct a longer distance entangled pair, this is one aim of this paper. We chose the Heisenberg XX open chain as the studying object, because it is simple and very useful in quantum information processing, such as it can be used for quantum computation [17, 18, 19, 20] and quantum communication [21, 22, 23].

In recently, the non-nearest pairwise entanglement in Heisenberg chain is also discussed [24, 25], the non-nearest entanglement comes from the introduce of magnetic field, which give the direct interaction between the non-nearest spin qubits, but they have two disadvantages, the first is its entanglement cannot be get the maximal value 1(the numerical results [24, 25] show that the non-nearest entanglement does not exceed 0.5), because the magnetic field only give the interaction between the z component of the non-nearest spin qubits; the second is that magnetic field will take complexness to the system and its application.

So we want to seek the relation between the pairwise entanglement and the system impurity, obviously, the system impurity is simpler than magnetic impurity, and maybe we can find the maximal non-nearest pairwise entanglement. This idea comes from our previous paper [26], in which we have introduced system impurity to three qubit Heisenberg XX spin ring and find that system impurity can control the nearest pairwise entanglement, so we suppose that the system impurity will help us to understand the non-nearest pairwise entanglement.

The main tool of discussing pairwise entanglement is the concept of entanglement of formation (EoF) and concurrence [27, 28], EoF can be used to measure the pairwise entanglement, concurrence C range from zero to one and it is monotonically relate to EoF, so concurrence is often used to measure the pairwise entanglement, we adopt this measurement in this paper and use C_{ij} to express the pairwise entanglement between the ith and jth qubits. $C_{ij} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$, where λ_k, $k = 1, 2, 3, 4$ are the square roots of the eigenvalues of the operator $\hat{\rho}_{ij} = \rho_{ij}(\sigma_i^y \otimes \sigma_j^y)\rho_{ij}^*(\sigma_i^y \otimes \sigma_j^y)$ in descending order, $\rho_{ij} = Tr_{non(ij)}\rho$ is the reduced density matrix of

the system.

In order to introduce the impurity, we write the Hamiltonian of the N-qubit Heisenberg XX open chain as following

$$H = \sum_{i=1}^{N-1} J_i J_{i+1}(\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-), \qquad (1)$$

where $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm \sigma^y)$, $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices, J_i is the contribution of the ith lattice to the exchange hopping and $J_i J_{i+1}$ is the exchange hopping between the ith and the $(i+1)$th lattice. If the ith lattice is impurity, then we let J_i as the impurity parameter, $J_k = 1$ $(k \neq i)$ are the normal parameters.

First we begin from the simplest case of the above Hamiltonian, i.e $N = 3$, different impurity can be seen from Fig. 1. The results of calculation show that J_1 or J_3 can make $C_{12} = 1$, J_2 can only make $(C_{12})_{max} = 0.457$, those can be understand easily, increasing J_1 or decreasing J_3 will couple stronger the first and the second qubits, i.e increasing their interaction and entanglement, while J_2 is an overall factor, in this sense this impurity system is equal to a normal system. Another main task of us is to discuss the relation between C_{13} and impurity parameter J_i, unfortunately, $C_{13} = 0$ no matter how we change the impurity parameter, which seems no confusion because there is no direct interaction between them.

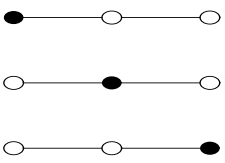

FIG. 1: The diagram of the impurity position in the 3-qubit open chain, where the solid circle is the impurity qubit, the hollow circle are the normal qubits. The site are labelled 1,2,3 from left to right.

Then let us see the next simplest case, i.e $N = 4$. As the impurity changing from the first to the forth site, we need to discuss the value of C_{12}, C_{23}, C_{13} and C_{14}. C_{12} and C_{23} are the nearest pairwise entanglement, C_{13} and C_{24} are the non-nearest pairwise entanglement. Different impurity can be seen from Fig. 2.

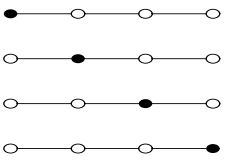

FIG. 2: The diagram of the impurity position in the four-qubit open chain, where the solid circle is the impurity qubit, the hollow circle are the normal qubits. The site are labelled 1,2,3,4 from left to right.

Tedious calculations show the following results:

(1) Changing every impurity site, increasing J_1 (or J_4) and decreasing J_2 (or J_3), can make $C_{12} = 1$, the physical meaning are clear: increasing J_1 (or J_4) and decreasing J_2 (or J_3) means that we isolate the first and the second qubits from the whole chain, their interaction is equal to the two qubit case, of cause it can get its maximal entanglement. (2) The maximal value of C_{23} is 0.457, the physical nature lie when increasing J_1 (or J_4) or decreasing J_2 (or J_3) means that we couple weaker the middle qubits, of course will decrease their entanglement; decreasing J_1(or J_4) or increasing J_2 (or J_3) means that we will take a qubit away from the whole chain, so the four-qubit chain become a three-qubit chain with uniform coupling, its maximal value is just 0.457. (3) $C_{13} = 0$ no matter how we change the impurity parameters, i.e there is no entanglement between the next nearest qubits, we still consider that there is no direct interaction between them. (4) There exist entanglement between the boundary qubits, its maximal value $(C_{14})_{max} = 0.457$, C_{14} increases as we decrease J_1 (or J_4) or increase J_2 (or J_3). Obviously, there is no direct interaction between the boundary quabits. No direct interaction theory seems invalidation. Certainly there exist another reason, which is just what we seek for.

No direct interaction theory can not give a harmonious explanation between $C_{13} = 0$ and $C_{14} \neq 0$, is there some different between them? Let us see this question between three-qubit and four-qubit cases, See Fig. 3.

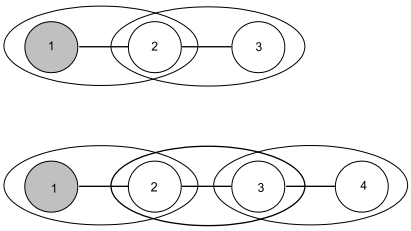

FIG. 3: The diagram of the comparison three-qubit and four-qubit open chain, where the solid circle is the impurity qubit, the hollow circle are the normal qubits, the ellipse denotes that there exist entanglement between the qubits in it.

The different is very clearly: there is only one qubit in the middle of the first and the third qubit, while two qubits (an entangled pair) in the first and the forth qubit. Obviously, the middle qubit(s) plays a key role in this different.

We suppose such an explanation: for three-qubit case, there is only one qubit in the middle, one qubit can not entangle with itself, although $C_{12} \neq 0$ and $C_{23} \neq 0$, $C_{13} = 0$, as if the entanglement is broken in the middle; while for four-qubit case, there is an entangled pair in the middle, $C_{12} \neq 0, C_{23} \neq 0$ and $C_{34} \neq 0$, so $C_{14} \neq 0$, i.e entanglement can be transferred through an entangled

pair.

For understanding more details about C_{14} we give a comparison among C_{12}, C_{23}, C_{34} and C_{14} through plotting the figures C_{ij} at fixed $J_1 = J$ (the first site is impurity) and fixed temperature, see Fig. 4.

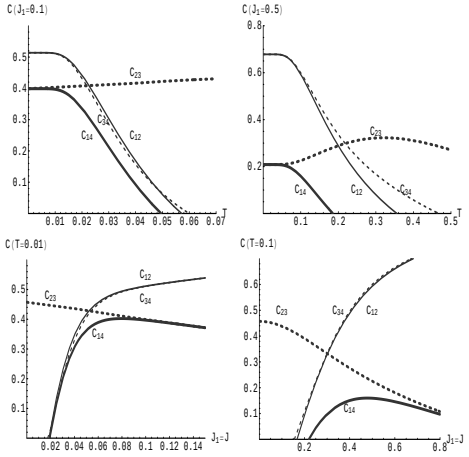

FIG. 4: The comparison among C_{12}, C_{23}, C_{34} and C_{14} at different fixed $J_1 = J$ and temperature.

From Fig. 4 we can see the transferred law of pairwise entanglement: C_{14} depends on the smallest of C_{12}, C_{34} and C_{23} ($\min(C_{12}, C_{34}, C_{23})$), when temperature is lower, C_{14} is equal to $\min(C_{12}, C_{34}, C_{23})$ approximately, when temperature is higher, C_{14} is less than $\min(C_{12}, C_{34}, C_{23})$ but with similarly changed trend. Changing the impurity site, similar results can be gotten. So if we want to find the maximal C_{14}, we must increase $\min(C_{12}, C_{34}, C_{23})$. Because C_{23} is more important than C_{12} and C_{34} so we call it "Entangled kernel".

We wander how C_{14} change if making entangled kernel $C_{23} = 1$. Considering the nearest interaction theory (for nearest pair, the stronger the interaction the bigger the pairwise entanglement), we need to make the interaction between the second and the third qubit strong, such a model is constructed in Eq. (1), let $N = 4, J_1 = J_4 = J, J_2 = J_3 = 1$, i.e the middle qubits are normal sites and the boundary qubits are impurity sites, as long as J is smaller than 1, the above condition is satisfied, obviously $C_{12} = C_{34}$ at this condition. Numerical results in some condition are shown in Fig. 5.

Fig. 5 shows that at lower temperature, C_{14} is only depended on C_{23} even $C_{12} = C_{34} = 0$ (but their interaction can not be too small). Choosing proper J and temperature, we can make $C_{14} = 1$. This is an important support for the importance of entangled kernel.

As a supporting for our theory, we calculate the pairwise entanglement in the five-qubit open chain with boundary impurities and find that $C_{13} = 0, C_{15} = 0$ and $C_{14} \neq 0$. For more than five qubit case, the calculation will be very difficult and tedious, we only construct a simple six-qubit case with $J_1 = J_6 = 0.1, J_2 = J_5 = 1, J_3 = J_4 = 10$, and get $C_{16} = 0.96098$ at the ground state(its entanglement is the maximal), of course we can construct longer distance maximal pairwise entanglement as long as the interaction strength decreases monotonously from middle to boundary.

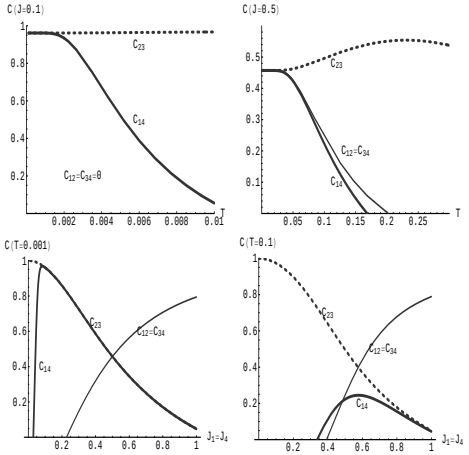

FIG. 5: The comparison among C_{12}, C_{23}, C_{34} and C_{14} at different fixed $J_1 = J_4 = J$ and fixed temperature.

In this paper, we calculate all the pairwise entanglement in the three and four-qubit Heisenberg open chain with system impurity, and find some interesting results: for the nearest pairwise entanglement, the stronger the interaction the bigger the entanglement; for the non-nearest pairwise entanglement, the condition of existing entanglement is decided by the qubit number in the middle, odd qubit number means no entanglement, even qubit number means the possibility of existing entanglement, this conclusion is supported by $C_{13} = 0$, $C_{15} = 0$ and $C_{14} \neq 0$; the attenuation of non-nearest pairwise entanglement (if exist) with temperature or impurity is quicker than the nearest pairwise entanglement, i.e. the transferred entanglement is more sensitive to the temperature and impurity.

The above conclusions are based on the open chain with qubit number no more than six, but they have generality in other qubit cases. First, pairwise entanglement is always connecting with double qubits, so its transfer must be depended on the entangled kernel, at this point they are coincidence; second, from the point of physics, the transfer of pairwise entanglement is the transfer of pairwise interaction, a best explanation is the example of four-qubit with boundary impurities.

These conclusions are very important for the solid system, through the transfer of entanglement we can construct pairwise entanglement with longer distance, we can increase this distance to micro-scale or even longer in theory, that is a practical scale for solid state quantum information. As we know there have at least two possible ways to realize such system: electrons floating on liquid helium and electron spins in coupled semicon-

ductor quantum dots.

We don't know if the transfer of entanglement in spin chain is still suitable for the entangled photons, because they have some different in the essence of production. If it is possible, we will no longer worry about the attenuation of entangled photons. We also don't know if we can use the non-maximal pairwise entanglement, if we can, at what value we can get the satisfied fidelity in quantum information processing?

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